THE QUARTIC OSCILLATOR IN CLASSICAL PHYSICS

The quartic oscillator, defined by the potential,

$$U(X) = \frac{1}{2}k_4 x^4$$

is a well-studied one-dimensional model problem in quantum mechanics. [1] This appendix presents an interesting classical analogue.

Consider a bead which can slide vertically on a frictionless wire, attached to two Hooke's law springs, as depicted in Fig. 1. [2] The potential energy is

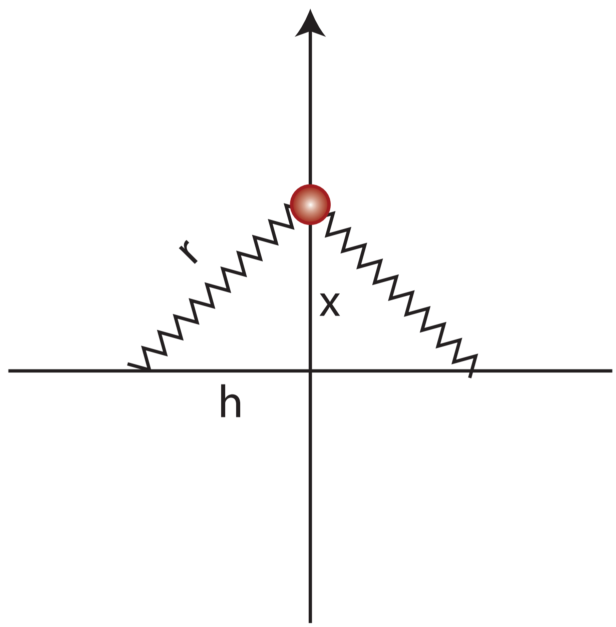

FIG. 1. Bead constrained to slide vertically along a frictionless wire

$$U(x, y) = 2\left[\frac{1}{2}k(r - r_0)^2\right]$$

The length of the spring is

$$r = \sqrt{h^2 + x^2}$$

Thus, the potential energy of the bead is (ignoring the effect of gravity)

$$U(x) = 2\left[\frac{1}{2}k(r - r_0)^2\right] = k\left(\sqrt{h^2 + x^2} - r_0\right)^2$$

Assume, further, that the rest length of the spring is $r_0 = h$. We can then expand the potential energy in a Maclaurin series in x, obtaining

$$U(x) = \frac{6k}{h^2}x^4 + \mathcal{O}(x^6)$$

Thus, near the equilibrium position, the motion of the bead is governed by the quartic oscillator potential, with quartic force constant $k_4 = 12k/h^2$.

At large excursions, however, the potential approaches that of a harmonic oscillator, since

$$\lim_{x\to\infty} U(x) = k \lim_{x\to\infty} \left(\sqrt{h^2 + x^2} - h\right)^2 = kx^2$$

A neat Java demonstration of the motion of the constrained bead is available online.

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[1] See, for example, R. P. McEachran, A. Rotenberg and M Cohen, J. Phys. B: Atom. Molec. Phys., **6**, L286 (1973) and references contained therein.

[2] We are grateful to Chris Jarzynski for his help with this analysis.